Math 10B with Professor Stankova
Worksheet, Midterm #1; Friday, 2/16/2018
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True/False

1. True False Among the problems we considered in class, a multi-stage process can be encoded (and solved) by either dependent choice or independent choice at each stage, or split into mutually exclusive cases.
2. True False We can turn any counting problem into a problem using the product rule or the sum rule.
3. True False Reversing the order of stages in a process does not affect the difficulty or efficiency of solving the problem.
4. True False $A \times B \times C$ for some sets A, B, C is another set made of all possible triplets (x, y, z) where x, y, z are any elements of the three sets.
5. True False We solved in class the problem of finding the size of the power of a set by setting up a multi-stage process with 2 independent choices at each stage.
6. True False The product rule for counting usually applies if we use the word "AND" between the stages of the process, while the sum rule for counting is usually used when we can finish the whole process in different ways/algorithms and we use the word "OR" to move from one way to another.
7. True False Among the problems we considered so far in class, a multi-stage process can be encoded (and solved) by either dependent choice or independent choice at each stage, split into mutually exclusive cases, or split into "good" and "bad" cases.
8. True False Counting problems where the phrase "at least once" appears may indicate using the complement, or equivalently, counting all cases and subtracting from them all "bad" cases.
9. True False Tree diagrams present a visual explanation of a situation, but unless one draw the full tree diagram to take into account all possible cases, the problem is not solved and will need more explanation/justification.
10. True False To find how many natural numbers $\leq n$ are divisible by d, we calculate the fraction n/d and round up in order to not miss any numbers.
11. True False We use 1 more than the ceiling (and not the floor) function in the statement of the Most General PHP because, roughly, we want to have one more pigeon that the ratio of pigeons to holes in order to "populate" a hole with the desired number of pigeons.

12. True False It is always true that $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ for any real number x, but equality of the two extreme terms of this inequality is never possible.

13. True False Proof by contradiction can be used to justify any version of the PHP.

14. True False A phrase of the type "at least these many objects" indicates what the pigeons should be in a solution with PHP, while "share this type of property" points to what the holes should be and how to decide to put a pigeon into a hole.

15. True False Erdos-Szekeres Theorem on monotone sequences is a generalization of the class problem on existence of an increasing or a decreasing subsequence of a certain length, and its proof assumes that one of two possibilities is not happening and shows that the other possibility must then occur.

16. True False Any version of the PHP implies existence of certain objects with certain properties and shows us how to find them.

17. True False To prove that there are some two points exactly 1 inch apart colored the same way on a canvas painted in black and white, it suffices to pick an equilateral triangle of side 1 in on this canvas and apply PHP to its vertices being the pigeons and the two colors (black and white) being the holes.

18. True False To show that a conclusion does not follow from the given conditions, we need to do more work than just show one counterexample.

19. True False A counterexample is a situation where the hypothesis (conditions) of a statement are satisfied but the conclusion is false.

20. True False The k-permutations of an n-element set are a special case of the k-combinations of this set.

21. True False An identity is an equality that is always true for any allowable values of the variables appearing in the equality.

22. True False An ordered k-tuple can be thought of some permutation of k elements, while an unordered k-tuple can be thought of a combination of k elements (perhaps, coming from a larger set).

23. True False To prove some identity combinatorially roughly means to count the same quantity in two different ways and to equate the resulting expressions (or numbers).

24. True False One good reason for 0! to be defined as 1 is for the general formula with factorials for C(n,k) to also work for k=0.

25. True False The number of combinations C(n,k) is the number of permutations P(n,k) divided by the number of permutations P(n,n).

26. True False The symmetry of permutations can be seen in the identity P(n,k)=P(n,n-k) for all integer $n, k \geq 0$.

27. True False The number of ways to split 10 people into two 5-person teams to play volleyball is $\frac{10!\cdot 10!}{2}$ because forgetting the 2 in the denominator would result in an overcount by a factor 2, which can be interpreted as an additional assignment of a court to each team on which to play (not required by the problem!).

28. True False It is possible to use Calculus to prove combinatorial identities.

29. True False Interpreting the same quantity in two different ways is not useful in proving binomial identities because, ultimately, one of the interpretations is harder (or impossible!) to calculate on its own.

30. True False The binomial coefficients appear in Pascal's triangle, as coefficients in algebraic formulas, and as combinations.

31. True False The alternating sum of the numbers in an even-numbered row of Pascal's triangle is zero for the simple reason that Pascal's triangle is symmetric across a vertical line; but the same statement for an odd-numbered row requires some deeper analysis since the numbers there do not readily cancel each other.

32. True False The basic combinatorial relation satisfied by binomial coefficients that makes it possible to identify all numbers in Pascal's triangle as some binomial coefficients can be written as $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ for $n, k \geq 1$.

33. True False The formula $1+2+3+\cdots+n = \binom{n+1}{2}$ for $n \geq 1$ is a special case of the Hockeystick Identity $\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k} = \binom{n+1}{k+1}$ for $n \geq k \geq 0$.

34. True False The binomial coefficients first increase from left to right along a row in Pascal's triangle, but then they decrease from the middle to the end of the row.

35. True False $k\binom{n}{k} = n\binom{n-1}{k-1}$ unless $k > n$.

36. True False The coefficient of x^3y^2 in $(x+y)^6$ is 0 because $2+3 \neq 6$; yet, it appears twice in the expanded form of $(x+y)^5$.

37. True False We can use the Binomial Theorem to prove all sorts of binomial identities, provided we recognize what x, y, and n to plug into it.

38. True False In general, it is harder to handle balls-into-boxes problems where the function must be surjective than where the function is injective or there are no restrictions on it.

39. True False The number of k combinations from n elements with possible repetition is $\binom{n+k-1}{n-1}$ and it matches the answer to the problem of distributing k identical biscuits to n hungry (distinguishable) dogs.

40. True False The number of 7-letter English words (meaningful or not, with possible repetition of letters) is not equal to the ways to distribute 7 equal bonuses to 26 people (with possible multiple-bonus winners).

41. True False The equation $x_1 + x_2 + x_3 + x_4 = 10$ in natural numbers has as many solutions as trying to feed 4 (different) dogs with 6 (identical) biscuits.

42. True False The expression $(x + y + z + t)^{2018}$ has $\binom{2020}{3}$ terms after multiplying through but before combining similar terms, and 4^{2018} terms after combining similar terms.

43. True False When we solve a problem one way, it is not useful to try to solve it in a second way because we already did the problem.

44. True False In general, it is harder to handle balls-into-boxes problems where the boxes are indistinguishable than where the boxes are distinguishable.

45. True False The number of ways to distribute b distinguishable balls into u distinguishable urns is $u!S(b,u)$ and the answer was obtained by solving first the same problem with indistinguishable urns and then labeling (or coloring) the urns to make them distinguishable.

46. True False The equation $x_1+x_2+x_3+x_4 = 10$ in natural numbers where the order of the variables does not matter has as many solutions as the number of ways to split 10-tuplets (10 identical kids) into 4 identical playpens, where each playpen has at least one kid.

47. True False We need to add several Stirling numbers of the second kind in order to count the ways to distribute distinguishable balls to indistinguishable boxes because all situations split into cases according to how many boxes are actually non-empty.

48. True False The direct formula for the Stirling numbers of the second kind can be derived using P.I.E., and this proof must be memorized in order to do well in this class.

49. True False We can prove the recursive formula for the Stirling numbers in a way very similar to the basic binomial identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ by selecting one special object and discussing the two possible cases from its viewpoint.

50. True False Two problem types in the 12-fold way table have extremely simple answers because an injective function cannot have a smaller domain than co-domain.

51. True False An algorithm is a finite sequence of well-defined steps that always lead to the correct (desired) output.

52. True False The Quick Sort algorithm is, on the average, faster than the Bubble Sort algorithm because the number of inversions in the list being sorted increases faster during the Quick Sort algorithm.

53. True False In class, we defined an inversion as two adjacent elements in a list that are out of order.

54. True False The maximal possible number of inversions in a list of 10 number is $P(10, 2) = 90$ while the minimal such number is 1.

55. True False A stable matching between n jobs and n people means that even if some person A prefers a job B on this list of jobs that he has not been given, the company that offers job B has hired another person C on it that they prefer to person A.

56. True False The number of roommate pairings among 2018 people can be written as $2017 \cdot 2015 \cdot 2013 \cdots 3 \cdot 1$ or alternatively also as $\frac{2018!}{2^{2018} 1009!}$.

57. True False It is never possible to pair up 2018 people into stable roommate pairs because, even if we manage to pair up 2014 of them in stable pairs, there will always be 4 of them to produce a counterexample of an impossible stable pairing.

58. True False To show that something is possible, it suffices to provide just one way of doing it, but to show that something is always true, we need to provide a proof that works for all cases.

59. True False The stable marriage algorithm produces the same final stable pairing, even if we reorder the "men" or if we switch the places of "men" and "women".

60. True False An argument by contradiction can be avoided if we are careful not to make mistakes in our proof.

61. True False Within MMI, the inductive step "If S_n is true then S_{n+1} is also true." implies that S_{n+1} is true.

62. True False When making the inductive hypothesis "Suppose S_n is true." we need to say "for some n "; yet, we cannot specify a particular number for n here.

63. True False At the end of a successful application of MMI, we conclude that S_n is true for some particular n's.

64. True False If we do not know the final precise answer to a problem, we cannot apply MMI until we conjecture what this answer is.

65. True False "Harry Potter is immortal." is not suitable for a proof by MMI, but it can be paraphrased into such a suitable statement.

66. True False It is never necessary to show the first several base cases in a proof by MMI; indeed, we do this just to boost our confidence in the truthfulness of the statement of the problem and we need to show only that the first base case is true.

67. True False Within the inductive step of a proof by MMI, we may occasionally need to use $S_{n-1}, S_{n-2}, S_{n-3}$, or some previous S_k (instead of S_n) in order to prove S_{n+1}.

68. True False Since the world will never end, the Tower of Hanoi problem for 64 initial discs on one of the three poles cannot be solved, whether by MMI or other methods.

69. True False Sending off newly-married couples to different honeymoon locations around the universe will provide a counterexample for the even version of the "Odd-pie fight" problem.

70. True False The "complement" property of probabilities, $P(\overline{A}) = 1 - P(A)$ for any $A \subseteq \Omega$, should be added to the definition of the probability space (Ω, P) because it is fundamental and always works.

71. True False When calculating the probability $P(A)$ for some event $A \subseteq \Omega$ on an "equally likely" finite probability space (Ω, P) , we can simply count the number of outcomes of A (the good possibilities) and divide that by all outcomes in Ω (all possibilities).

72. True False It is incorrect to say that the elements of Ω are "outcomes" since they are actually inputs of the probability P, and not outputs.

73. True False The probability function P is defined as $P : \Omega \to [0, 1]$ such that $P(\Omega) = 1, P(\emptyset) = 0$, and $P(A \cup B) = P(A) + P(B)$ for any disjoint subsets A and B of Ω.

74. True False The formula $\left\lceil \frac{N}{d} \right\rceil$ appears when calculating the probability of a natural number $n \leq N$ to be divisible by d.

75. True False MMI is not really necessary to formally prove the property $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$ for pairwise disjoint A_i's because the property is quite intuitive and, to prove it, we can just apply over and over again the basic property of probabilities $P(A \cup B) = P(A) + P(B)$ for various non-overlapping $A, B \subseteq \Omega$.

76. True False If we experiment with throwing two fair dice and adding up the two values on the dice, and if we decide to represent the outcome space Ω as the set all of possibilities for the sum; i.e., $\Omega = \{2, 3, \ldots, 12\}$, then the corresponding probability P will not be the "equally likely" probability, making us reconsider the choice of the outcome space Ω in the first place.

77. True False The property $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any $A, B \subseteq \Omega$ is true for any probability space (Ω, P) , but it can be proven using baby P.I.E. only when P is the "equally likely" probability on a finite outcome space Ω , while a more general argument is needed for other (Ω, P).

78. True False The "defective dice" problem from the Discrete Probability handout does not have a unique solution; i.e., there is another second die that can yield, together with the original defective die, the same probabilities for the sums of the values on the two dice as two normal fair dice.

79. True False In the Monty Hall Problem with n doors (for any $n \geq 3$) we should switch doors (after the host opens a non-winning door) because with this strategy the probability of winning is $\frac{n-1}{n(n-2)}$; however, for $n = 2$ this formula makes no sense and, moreover, half of the time the game itself is impossible to complete as designed for $n = 2$.

80. True False It may not be possible to calculate $P(A \cap B)$ using just $P(A)$ and $P(B)$, but if we also know one of $P(A|B)$ or $P(B|A)$, we can do it!

81. True False $P(A|B)$ can never be equal to $P(B|A)$ unless $P(A) = P(B)$.

82. True False The formula $P(A) = P(A|B) \cdot P(B) + P(A|\overline{B}) \cdot P(\overline{B})$ works for any events $A, B \subseteq \Omega$, as long as $P(B) > 0$.

83. True False Despite the suggestive notation, the conditional probability $P(A|B)$ was originally defined through a formula and we had to prove that it indeed is in $[0, 1]$ in order to consider $P(A|B)$ as an actual probability.

84. True False To prove the Probability "Baby P.I.E." property $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, one could split $A \cup B$ on the LHS into three disjoint subsets, similarly split on the RHS A and B each into two subsets, cancel, and match the resulting probabilities on the two sides.

85. True False When considering conditional probability, we are restricting the original outcome space Ω to a smaller subspace B given by the condition of something having happened.

86. True False When $A \subset B$, the conditional probability $P(A|B)$ can be expressed as the fraction $\frac{P(A)}{P(B)}$ (given all involved quantities are well-defined).

87. True False Bonferroni's inequality $P(E \cap F) \geq p(E) + P(F) - 1$ is, in disguise, the well-known fact that $P(E \cup F) \leq 1$.

88. True False When selecting at random two cards from a given 6-card hand (from a standard deck) that is known to contain 2 Kings, it is more likely to end up with at least one King than no King.

89. True False When selecting at random two cards from a given n-card hand (from a standard deck) that is known to contain 2 Kings, the smallest n for which it is more likely to end up with no King than with at least one King is $n = 8$.

1 Problems

90. How many ways can you rearrange the letters in BERKELEY?

91. There are 72 students trying to get into 3 of my sections. There are $27, 20, 25$ openings respectively. How many ways are there for these students to enroll?

92. How many ways can I put 20 Tootsie rolls into 5 goodie bags so that each goodie bag has at least 2 Tootsie roll?

93. Show that when you place 9 coins on an 8×10 boards, at least two coins must be on the same row.

94. How many license plates with 3 digits followed by 3 letters do not contain the both the number 0 and the letter O (it could have an O or a 0 but not both).

95. Prove that $\sum_{k=0}^{n} 5^k \binom{n}{k} = 5^0 \binom{n}{0} + 5^1 \binom{n}{1} + \cdots + 5^n \binom{n}{n} = 6^n$.

96. How many ways can I split up 30 distinguishable students into 6 groups each of size 5?

97. Find a formula for $1 + 2 + 4 + \cdots + 2^n$ and prove it.

98. How many 5 digit numbers have strictly increasing digits (e.g. 12689 but not 13357).

99. How many 5 digit numbers have increasing digits (you can have repeats e.g. 12223 or 22222)?

100. A 7 phone digit number $d_1d_2d_3 - d_4d_5d_6d_7$ is called memorable if $d_1d_2d_3 = d_4d_5d_6$ or $d_1d_2d_3 = d_5d_6d_7$. How many memorable phone numbers are there?

101. How many people do you need in order to guarantee that at least 3 have the same birthday?

102. How many 5 letter words have at least two consecutive letters are the same?

103. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.

104. How many ways can I put 60 seeds in 10 indistinguishable boxes so that each box has at least 3 seeds?

105. I have 4 indistinguishable blue coins and 4 indistinguishable gold coins. How many ways can I stack them?

106. When I go to CREAM, I order 4 scoops of ice cream out of 10 possible flavors (I can get more than one scoop of a flavor)?

107. Show that in a class of 30 students, there must exist at least 10 freshmen, 8 sophomore, 8 juniors, or 7 seniors.

108. How many ways can I buy 24 donut holes if there are 8 different flavors?

109. How many ways can I split 200 indistinguishable donut holes into 8 non-empty bags? (The bags are indistinguishable)

110. How many ways can I split 200 indistinguishable donut holes into at most 8 bags? (The bags are indistinguishable)

111. I have 5 identical rings that I want to wear at once. How many ways can I put them on my hand (10 fingers) if each ring must go on a different finger?

112. Suppose that Alice chooses 4 distinct numbers from the numbers 1 through 10 and Bob chooses 4 numbers as well. What is the probability that they chose at least one number in common?

113. How many positive integers less than $10,000$ have digits that sum to 9?

114. I am planting 15 trees, 5 willow trees and 10 fir trees. How many ways can I do this if the two willow trees cannot be next to each other?

115. How many ways can we put 5 distinct balls in 20 identical bins?

116. How many ways can I distribute 30 Snickerdoodle cookies and 20 chocolate chip cookies to 25 students if there is no restriction on the number of cookies a student gets (and some students can get none)?

117. What is the coefficient of the term $a^{10}b^{20}c^{30}$ in $(2x + 3y + 4z)^{60}$?

118. What is the probability that a roll a sum of 9 with two dice given that I rolled a 6?

119. In the US (300 million people), everyone is a male or female and likes one of 10 different colors. Show that there exist at least 3 people that have the same gender, like the same color, have the same three letter initial, and have the same birthday?

120. I roll two die. What is the probability that I roll a 2 given that the product of the two numbers I rolled is even.

121. I have 5 identical rings that I want to wear at once. How many ways can I put them on my hand (10 fingers) if it is possible to put all 5 on one finger?

122. How many ways can I split up 30 distinct students into 6 non-empty groups?

123. What is the probability that in a hand of 5 cards out of a deck of 52 cards, there is a pair of aces given that there is an ace?

124. How many ways can I plant 15 trees in 5 different yards if each yard has to be nonempty?

125. In a bag of coins, 2/3 of them are normal and 1/3 have both sides being heads. A random coin is selected and flipped and the outcome is a heads. What is the probability that it is a double head coin?

126. How many numbers less than or equal to 1000 are not divisible by $2, 3$, or 5?

127. How many ways can you line up 4 couples if each couple needs to stand next to each other?

128. Prove that $2 + 4 + \cdots + 2n = n(n + 1)$ for all $n \geq 1$.

129. How many ways can I put 30 Snickerdoodle cookies and 20 chocolate chip cookies into 10 identical bags so that each bag has at least one Snickerdoodle cookie and one chocolate chip cookie?